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A Property of the Zeros of the Legendre Polynomials

CARL C. GROSJEAN

Seminarie voor Wiskundige Natuurkunde, Rijksuniversiteit-Gent, B-9000 Gent, Belgium

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In this paper, it is proven that the zeros of the Legendre polynomials $P_n(x)$ satisfy the inequality

$$(1-x_{j-1}^{(n)})(1-x_{j+1}^{(n)})<(1-x_j^{(n)})^2, \qquad \forall j\in\{2,3,...,n-1\}, \forall n\in\{3,4,...\}.$$

This result is obtained by applying Sturm's comparison theorem to two homogeneous linear differential equations of second order, each of which has a particular solution deduced from the function

$$[x(2-x)]^{1/2}P_n(1-x), \qquad 0\leqslant x\leqslant 2.$$

1. INTRODUCTION

During the open problems session of the Laguerre Symposium on Orthogonal Polynomials and Their Applications which took place at Bar-le-Duc (France) in October 1984, P. G. Nevai posed the following problem originating with R. DeVore:

Prove that for each n belonging to $\{3, 4,...\}$, the zeros of the Legendre polynomial $P_n(x)$, arranged in ascending order between the bounds -1 and 1, namely,

$$-1<x_1^{(n)}<x_2^{(n)}<\cdots<x_n^{(n)}<1, \tag{1}$$

satisfy

$$(1-x_{j-1}^{(n)})(1-x_{j+1}^{(n)})\leqslant(1-x_j^{(n)})^2, \qquad \forall j\in\{2,3,...,n-1\}. \tag{2}$$

This problem is related to questions of monotonicity, i.e., whether or not the second and higher order differences of the sequences of consecutive

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positive zeros of certain classical orthogonal polynomials, arranged in ascending order, are all positive [2–5]. Indeed, once the strict inequality formulated in the abstract is proven, it appears that

$$\ln(1-x_{j-1}^{(n)}) - 2\ln(1-x_j^{(n)}) + \ln(1-x_{j+1}^{(n)}) < 0,$$
$$\forall j \in \{2, 3, \ldots, n-1\}, \forall n \in \{3, 4, \ldots\},$$

showing that $\ln(1-x_k^{(n)})$, $k=1, 2, \ldots, n$, is a convex function of k. Here, the differences of second order are all negative.

A quick numerical verification carried out for $n=3$, 4, and 5 confirms the assertion (2) without the equality sign. In the analytical proof which will follow, I temporarily omit the superscript in the notation of the zeros of $P_n(x)$ for the sake of simplicity. This proof will consist of an application of Sturm's comparison theorem in the formulation of Ahmed *et al.* [1]. I recall this theorem here for convenience, using my own notation:

Let the real function $y(x)$ be a non-trivial solution of the differential equation

$$y''(x) + f(x)\, y(x) = 0, \qquad x \in \mathbb{R}, \tag{3}$$

having $\alpha_1, \alpha_2, \ldots, \alpha_m$ as consecutive zeros in the real interval $]a, b[$:

$$a < \alpha_1 < \alpha_2 < \cdots < \alpha_m < b.$$

Similarly, let $z(x)$ be a non-trivial solution of

$$z''(x) + g(x)\, z(x) = 0 \tag{4}$$

with consecutive zeros $\beta_1, \beta_2, \ldots, \beta_m$ belonging to $]a, c[$:

$$a < \beta_1 < \beta_2 < \cdots < \beta_m < c.$$

Suppose that the coefficients $f(x)$ and $g(x)$ are continuous functions satisfying $f(x) < g(x)$ in $[a, \beta_m]$, and that $y(x)$ and $z(x)$ are such that

$$\lim_{x \to a+0} [y'(x)\, z(x) - z'(x)\, y(x)] = 0.$$

Then

$$\beta_k < \alpha_k, \qquad \forall k \in \{1, 2, \ldots, m\}. \tag{5}$$

Under the conditions of Sturm's theorem, the differential Eq. (4) is said to be a Sturmian majorant of the differential Eq. (3).

2. Proof of Assertion (2) (without Equality Sign)

Consider the real function

$$u_n(x) = \sqrt{x(2-x)}\, P_n(1-x), \qquad 0 \leqslant x \leqslant 2,$$

which has its n (single) zeros lying between the branch-points 0 and 2 of its extension to $\mathbb{C}$:

$$0 < 1 - x_n < 1 - x_{n-1} < \cdots < 1 - x_1 < 2,$$

whereby the x-values are the zeros of $P_n(x)$ as mentioned in (1). It is a solution of the differential equation

$$U''(x) + \left[\frac{n(n+1)}{x(2-x)} + \frac{1}{x^2(2-x)^2}\right] U(x) = 0.$$

Let us choose one of the zeros of $P_n(x)$, x_r say, and put

$$x = (1 - x_r)\, t, \qquad v_n(t) = u_n((1 - x_r)\, t), \qquad 0 \leqslant t \leqslant \frac{2}{1 - x_r}. \tag{6}$$

The n zeros of $v_n(t)$ are comprised in the inequalities

$$0 < \frac{1 - x_n}{1 - x_r} < \frac{1 - x_{n-1}}{1 - x_r} < \cdots < \frac{1 - x_{r+1}}{1 - x_r} < 1 < \frac{1 - x_{r-1}}{1 - x_r} < \cdots$$
$$< \frac{1 - x_1}{1 - x_r} < \frac{2}{1 - x_r}, \tag{7}$$

and $v_n(t)$ is a particular solution of the differential equation

$$V''(t) + \left\{\frac{n(n+1)(1 - x_r)}{t[2 - (1 - x_r)\, t]} + \frac{1}{t^2[2 - (1 - x_r)\, t]^2}\right\} V(t) = 0. \tag{8}$$

We shall regard this equation as the counterpart of (3), with $V(t)$ playing the role of $y(x)$.

Next, we wish to repeat this way of proceeding, with r replaced by $r - 1$. This requires that r be restricted to the set of integers $\{2, 3, ..., n\}$. Then we can define the function

$$w_n(t) = u_n((1 - x_{r-1})\, t), \qquad 0 \leqslant t \leqslant \frac{2}{1 - x_{r-1}}, \tag{9}$$

whose n zeros satisfy

$$0 < \frac{1 - x_n}{1 - x_{r-1}} < \frac{1 - x_{n-1}}{1 - x_{r-1}} < \cdots < \frac{1 - x_r}{1 - x_{r-1}} < 1 < \frac{1 - x_{r-2}}{1 - x_{r-1}} < \cdots$$
$$< \frac{1 - x_1}{1 - x_{r-1}} < \frac{2}{1 - x_{r-1}} \left(< \frac{2}{1 - x_r} \right). \tag{10}$$

$w_n(t)$ is a particular solution of the differential equation

$$W''(t)+\left\{\frac{n(n+1)(1-x_{r-1})}{t[2-(1-x_{r-1})\,t]}+\frac{1}{t^2[2-(1-x_{r-1})\,t]^2}\right\}W(t)=0. \quad (11)$$

We regard this differential equation as the counterpart of (4), with $W(t)$ replacing $z(x)$. The formulae from (6) to (11) are meaningful for any r belonging to $\{2, 3,..., n\}$. Now, the two functions of t between the braces appearing in (8) and (11), respectively, are continuous in $0<t<2/(1-x_{r-1})$. In this interval, we have that

$$\begin{aligned}&\frac{n(n+1)(1-x_r)}{t[2-(1-x_r)\,t]}+\frac{1}{t^2[2-(1-x_r)\,t]^2}\\ &\quad<\frac{n(n+1)(1-x_{r-1})}{t[2-(1-x_{r-1})\,t]}+\frac{1}{t^2[2-(1-x_{r-1})\,t]^2}\end{aligned} \quad (12)$$

is equivalent to

$$-2n(n+1)[2-(1-x_{r-1})t][2-(1-x_r)t]<4-(1-x_{r-1})\,t-(1-x_r)t.$$

The latter inequality is fulfilled in $0<t<2(1-x_{r-1})$ since the left-hand side is negative and the right-hand side is positive. Hence, (12) holds a fortiori in the interval

$$\left[1, \frac{1-x_1}{1-x_{r-1}}\right], \quad (13)$$

which plays the role of $[a, \beta_m]$ mentioned in Section 1, is non-degenerate for r still more restricted than before, namely, $r\in\{3, 4,..., n\}$, whereby $n\geqslant 3$ and is embedded in $0<t<2/(1-x_{r-1})$, according to (10).

Finally, with the lower bound a equal to 1, there comes

$$\lim_{t\to 1}\,[v_n'(t)\,w_n(t)-w_n'(t)\,v_n(t)]=0,$$

since $v_n(t)$, $w_n(t)$, $v_n'(t)$, $w_n'(t)$ are continuous in the neighbourhood of $t=1$, $v_n(1)=u_n(1-x_r)=0$, $w_n(1)=u_n(1-x_{r-1})=0$, and $v_n'(1)$, $w_n'(1)$ are finite. Therefore Eq. (11) is a Sturmian majorant of Eq. (8) in the interval (13), and we conclude that

$$\frac{1-x_{r-2}}{1-x_{r-1}}<\frac{1-x_{r-1}}{1-x_r}, \qquad \forall r\in\{3, 4,..., n\}, \forall n\in\{3, 4,...\}, \quad (14)$$

according to (5) applied for $k=1$. Replacing $r-1$ by j in (14) yields

$$(1-x_{j-1}^{(n)})(1-x_{j+1}^{(n)})<(1-x_j^{(n)})^2,$$
$$\forall j\in\{2,3,\ldots,n-1\},\ \forall n\in\{3,4,\ldots\},\tag{15}$$

which is the assertion (2) without equality sign.

It may be asked whether the full scale application of Sturm's comparison theorem, i.e., with $k>1$ in (5), leads to any additional results concerning the zeros of the Legendre polynomials. The answer is negative, because (5) when applied to (7) and (10) leads to

$$\frac{1-x_{j-1}^{(n)}}{1-x_{r-1}^{(n)}}<\frac{1-x_j^{(n)}}{1-x_r^{(n)}},\qquad \forall j\in\{2,3,\ldots,r-1\},$$
$$\forall r\in\{3,4,\ldots,n\},\ \forall n\in\{3,4,\ldots\}.$$

Completely equivalent to this is

$$\frac{1-x_{j-1}^{(n)}}{1-x_j^{(n)}}<\frac{1-x_{r-1}^{(n)}}{1-x_r^{(n)}},\qquad \forall r\in\{j+1,j+2,\ldots,n\},$$
$$\forall j\in\{2,3,\ldots,n-1\},\ \forall n\in\{3,4,\ldots\},\tag{16}$$

which is in essence nothing but the result (15) when it is rearranged as a chain of consecutive inequalities of ratios. From (16), it follows that

$$(1-x_{j-1}^{(n)})(1-x_r^{(n)})<(1-x_{r-1}^{(n)})(1-x_j^{(n)}),\tag{17}$$

valid for

$$\forall r\in\{j+1,j+2,\ldots,n\},\qquad \forall j\in\{2,3,\ldots,n-1\},\qquad \forall n\in\{3,4,\ldots\},\tag{17$'$}$$

but all these inequalities are a consequence of (15).

References

1. S. Ahmed, A. Laforgia, and M. E. Muldoon, On the spacing of the zeros of some classical orthogonal polynomials, *J. London Math. Soc.* (2) **25** (1982), 246–252.
2. L. Lorch and P. Szegö, Higher monotonicity properties of certain Sturm–Liouville functions, *Acta Math.* **109** (1963), 55–73.
3. L. Lorch and P. Szegö, Higher monotonicity properties of certain Sturm–Liouville functions. II, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys.* **11** (1963), 455–457.
4. L. Lorch, M. E. Muldoon, and P. Szegö, Higher monotonicity properties of certain Sturm–Liouville functions. III, *Canad. J. Math.* **22** (1970), 1238–1265.
5. L. Lorch, M. E. Muldoon, and P. Szegö, Higher monotonicity properties of certain Sturm–Liouville functions. IV, *Canad. J. Math.* **24** (1972), 349–368.

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